Correction made on Page 7 V.C. 4-18-50

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METHODS OF TREATMENT OF DISPLACEMENT INTEGRAL EQUATIONS

Recipe Book

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METHODS OF TREATMENT OF DISPLACEMENT INTEGRAL EQUATIONS

Recipe Book

In LA-53 a detailed treatment is given of the general theory of displacement integral equations. It is the purpose of this summary to present in concise form the methods of treatment and recipes for the solution of such problems. The pertinent numerical data is presented in the form of graphs appended to this summary. A few of these graphs are of no significance to a person reading only this summary, but they are included for the use of readers of the main text. These graphs are numbered, beginning with Fig. V. The first four figures are included in the text of LA-53 and are not pertinent in this summary.

The type of integral equation with which we deal is

$$n(\underline{r}) = \int d\underline{r}' \, n(\underline{r}') \, K(|\underline{r} - \underline{r}'|) \, F(\underline{r}').$$

Here the kernel K is a function of the distance between the two points. The weighting function, $F(\underline{r}')$, serves the purpose of defining the range of integration and the properties of the material in the various regions considered. Equations of this type have application in many problems involving the multiplying and diffusion of neutrons ~~in~~ fissionable, scattering, and absorbing materials.

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One of the important problems of this project is that of the critical sizes and multiplication rates of gadgets made of such materials that the neutrons may be considered monoenergetic. If the scattering is treated as isotropic and all parts of the gadget have the same mean free path, the integral equation is

$$n(\underline{r}) = \int d\underline{r}' \, \frac{1 + f(\underline{r}')}{1 + \gamma} \, n(\underline{r}') \, \frac{e^{-|\underline{r} - \underline{r}'|}}{4\pi(\underline{r} - \underline{r}')^2} \qquad (1)$$

γ is the multiplication rate in such units that e^{γ} is the multiplication in a mean free time. Thus the time dependence is $e^{\gamma v N \sigma_t t}$ where σ_t is the total collision cross section $\sigma_s + \sigma_c + \sigma_f$, N the nuclear density and v the neutron velocity. The unit of length in equation (1) is the "mean attenuation distance", the mean free path, λ, divided by $(1 + \gamma)$, i.e. $1/\left[N\sigma_t(1+\gamma)\right]$. The factor $(1 + f)$ represents the mean number of neutrons emerging from each collision. f is zero for a pure scatterer, -1 for a pure absorber, and positive for active material. $f = \left[(\nu - 1)\sigma_f - \sigma_c\right]/\sigma_t$ where ν is the mean number of neutrons emerging from a fission process.

Special studies have been made to determine the effect of the failure of our assumption of isotropic scattering and a single neutron energy (cf LA-53, Chapter VI). These studies indicate that, for moderate anisotropy and velocity spread, a good approximation is to take the present formalism, where σ_s is the transport average, $\overline{(1 - \cos\theta)\sigma_s(\theta)}$, of the scattering cross section and v is the harmonic average of the velocities actually occurring

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$1/(\overline{1/v})$. This average cross section and velocity must be used consistently throughout the treatment, i.e., it must be used in the definition of f and γ as well as in the determination of the scale of length.

In a region far removed from any boundary or interface, i.e. in the deep interior of a large region of constant f the integral equation can be simplified by extending the range of integration to infinity (and assuming that $n(r)$ remains regular to infinity). The integral equation then reduces to a differential equation of the same form as the familiar diffusion equation

$$(\Delta + k^2)\, n(r) = 0 \qquad (2)$$

where k is determined by the condition

$$(\tan^{-1} k)/k = (1 + \gamma)/(1 + f) \qquad (3)$$

The unit of length, which occurs in the Laplacian, is still the mean attenuation distance. It will be observed that although this equation (2) is of the same form as the diffusion equation there is in general no simple relationship between the two values of the constant entering. The two values do coincide for very small values of <u>both</u> f and γ, the limit in which diffusion theory must be correct. The smallness of k^2, hence of $f - \gamma$, is not sufficient to ensure agreement.

The wave equation, (2), has simple well-known solutions, $\sin \underline{k} \cdot \underline{r}$ for a plane problem, $\sin kr/r$ for spherical symmetry, $J_0(kr)$ for cylindrical

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symmetry, etc. Which of these solutions is to be used is in most cases immediately indicated by the symmetry of the boundaries.

The magnitude of k is determined by equation (3). A graph of the function $(\tan^{-1} k)/k$ and its reciprocal occurs in Fig. V.

In a plane problem, i.e. one in which $f(\underline{r})$ and $n(\underline{r})$ depend only on the cartesian coordinate x, the integral equation (1) reduces to

$$n(x) = \int dx' \frac{1 + f(x')}{2(1+\gamma)} \; n(x') \, E(|x - x'|) \qquad (4)$$

where

$$E(|y|) = \int_{|y|}^{\infty} \frac{ds e^{-s}}{s}$$

which is tabulated under another symbol in the first section of Janke-Emde, e.g. The asymptotic solution of (4) far from a boundary is $A \sin\left[k(x + x_0)\right]$ where k is determined by condition (3). If the medium is homogeneous throughout all space, x_0 and A are completely arbitrary. If the medium is homogeneous but extends only to one side of a plane boundary, say at $x = 0$, then A is still arbitrary but x_0 is determinate. The solution will not be exactly $A \sin\left[k(x + x_0)\right]$ near the boundary but will approach this form as an asymptote far from the boundary. (See Fig. XI.) The exact solution, $n(x)$, is found in LA-53, Chapter III, and the phase of the asymptotic sine solution, x_0, determined. This asymptotic solution, $A \sin\left[k(x + x_0)\right]$, vanishes at $x = -x_0$, i.e. at a distance x_0 beyond the boundary. It should be

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emphasized that the point, $x = -x_0$, is not a root of the true solution but of the asymptotic solution, $A \sin\left[k(x + x_0)\right]$, which is approached by the true solution for large x. For this reason, the distance, x_0, is known as the "extrapolated end-point". x_0, like x itself, is measured in terms of the mean attenuation distance, $\lambda/(1 + \gamma)$.

x_0 times $(1 + f)/(1 + \gamma)$ is nearly constant, having values slightly greater than .71 over a considerable range. A graph of this product is given in Fig. VI.

Intorduction of Tamper

Another problem which can be solved exactly is that of the neutron distribution in two homogeneous media of the same mean free path on the two sides of a plane boundary at $x = 0$. f may be different on the two sides of the boundary, e.g., zero or negative on one side (tamper), positive on the other (active material). In this case there will be asymptotic solutions far from the boundary on either side. These two asymptotic solutions will be the solutions of equation (2) with the two values of k^2 determined by the two values of f. The solution in the tamper may be hyperbolic (exponential) instead of sinusoidal. This will occur if, in the tamper, $(1 + f)/(1 + \gamma)$ is less than 1, thus giving an imaginary solution for equation (3). In this event the two equations (2) and (3) are more conveniently replaced by

$$(\Delta - k^2)\, n(\underline{r}) = 0 \qquad (2')$$

and

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$$(\tanh^{-1} k)/k = (1 + \gamma)/(1 + f) \qquad (3')$$

Graphs of $(\tanh^{-1} k)/k$ and its reciprocal are given in Fig. V.

The boundary condition has been evaluated for the case of an exponentially decaying asymptotic tamper solution (decaying exponentially away from the boundary). The extrapolated end point, x_o, of the asymptotic core solution $A \sin\left[k(x + x_o)\right]$, is found to be the difference between two terms, one simple and one small. The simple term is

$$(1/k_o) \tan^{-1} (k_c/k_t)$$

where k_c is the wave-number of the asymptotic solution in the core (active material), $\sin k_c(x + x_o)$, and k_t the (hyperbolic) wave-number of the asymptotic solution in the tamper, $e^{k_t x}$. This term is exactly the value of x_o which would be obtained by applying the diffusion-theoretic boundary condition, i.e. the continuity at the boundary of the logarithmic derivative of $n(x)$, to the two asymptotic solutions. The small term is denoted by Δx_o and must be subtracted from this simple term to give the true value of x_o. Δx_o and $\Delta x_o (1 + f_t)/(1 + \gamma)$ are presented in Figs. VII and VIII. It will be seen from Fig. VIII that this "extrapolated end-point correction" if measured in true mean free paths has approximately the value $.045/(1 + f_t)$. This approximation is independent of γ and is a sufficiently good one since the corrections give a very small contribution to the dimensions of the gadget. It is reasonable to assume that if the asymptotic solution in the tamper is

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not a pure decaying exponential but, e.g., a hyperbolic sine or cosine the true end-point, x_o, will again be the diffusion-theoretic end-point, obtained by equating logarithmic derivatives, less the Δx_o of Figs. VII and VIII.

Solution for Spheres

It is shown in LA-53, Chapter II, that there exists an exact parallelism between problems of spherical symmetry and corresponding problems of plane symmetry. The correspondence is as follows: The problem of plane symmetry is that of a series of infinite slabs of finite thickness of the materials occurring in the spherical problem. The sequence of materials and slab thicknesses is such that a line perpendicular to the plane faces passes through the same sequence of materials and for the same distances as are found along a diameter of the sphere. This "slab problem" thus has reflection symmetry about its mid-section. The integral equation for this distribution of matter will thus possess solutions of both odd and even symmetry. The odd solutions, $n_o(x)$, (x measured from the mid-section) are directly related to all of the solutions of the spherical problem, $n_s(r)$.

$$n_o(x) = \left[r\, n_s(r) \right]_{r=x}$$

This relationship between spherical and plane problems enables us to carry over directly the results of the end-point analysis to the treatment of spherical problems.

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An example of the use of this relationship is the treatment of the untamped sphere. The fundamental solution in an untamped sphere is related to the first odd solution in a slab of thickness equal to the diameter of the sphere, say 2a. The "asymptotic solution" in the slab is then $\sin k x$ where x is again measured from the mid-section. This asymptotic solution must vanish a distance x_0 beyond the boundary, i.e. at $x = a + x_0$, where x_0 is that given by Fig. VI. The sine function first vanishes for argument π, thus

$$k(a + x_0) = \pi$$

k and x_0 are determined by f and γ, hence also a. The only assumption made here is the assumption that in the middle of the slab there exists a region in which the asymptotic solution is well established so that the two boundary conditions (at a and $-a$) can be applied without interfering with each other. A comparison of the results of this calculation with those of variation theory shows this assumption to be completely justified throughout the useful range. (Cf. Fig. XVI.)

The same method is applied to the tamped sphere. The tamped-sphere problem is replaced by that of a three-slab "sandwich". The center slab is of core material and of thickness equal to the core diameter, the two outer slabs of tamper material and of thickness equal to the tamper thickness. The solution sought is the first solution which is odd about the middle of the sandwich. The most convenient method of solution is to take definite values

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for the tamper thickness, for γ, and for the two f-values. The solution then proceeds from the outside toward the center, determining finally the core thickness, hence the core radius in the spherical analog. The asymptotic tamper solution in the sandwich problem will be a sine or hyperbolic sine vanishing at a distance x_o beyond the tamper surface. The wave-number and x_o are determined by f_t and γ. These two quantities, together with the tamper thickness, determine the phase, hence also the logarithmic derivative of the asymptotic tamper solution at the core-tamper interface. This logarithmic derivative is equated to the logarithmic derivative of the asymptotic core-solution, $\sin k_c x$. The value of x so determined will be less than the true half-thickness of the core by an amount Δx_o. This true half-thickness of the core is equal to the radius of the core in the spherical problem.

Extensions of the End-Point Method

The asymptotic solution vanishes at a distance x_o beyond an untamped surface. This has been shown by the end-point theory to hold both for plane and spherical surfaces. It has been found empirically (by comparison with variation results) to hold also for cylindrical surfaces. This result is not surprising, since a cylindrical surface is in a sense midway between the plane and spherical surfaces, but is not in any way guaranteed by the end-point theory. It is assumed that the same gratuitous continuation of the validity of the end-point boundary condition holds for a tamped cylindrical surface. This assumption is supported by rough numerical results.

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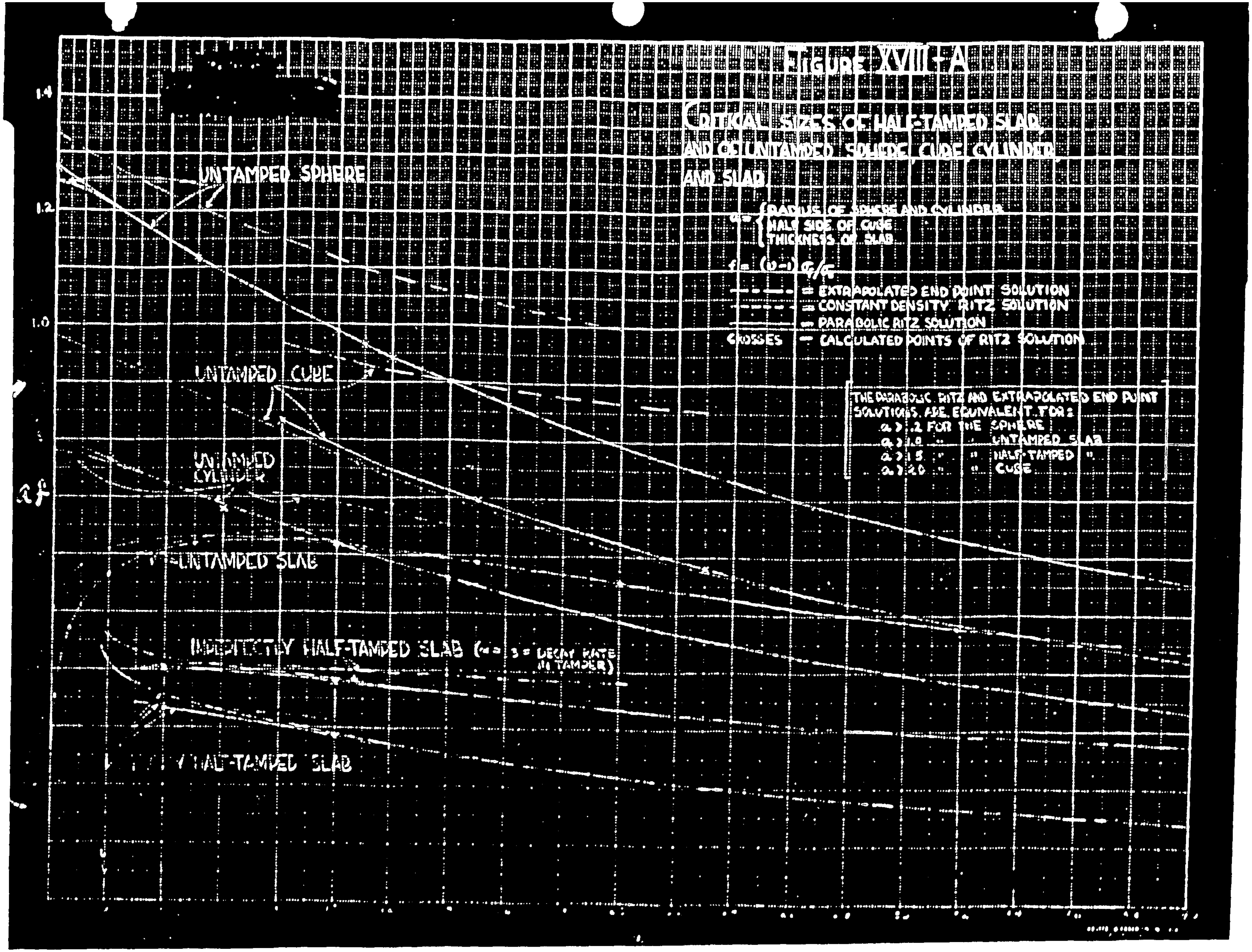


Figure XVIII-A
Critical sizes of half-tamped slab
and of untamped sphere, cube, cylinder,
and slab
RADIUS OF SPHERE AND CYLINDER
HALF SIDE OF CUBE
THICKNESS OF SLAB
= EXTRAPOLATED END POINT SOLUTION
= CONSTANT DENSITY RITZ SOLUTION
= PARABOLIC RITZ SOLUTION
CROSSES = CALCULATED POINTS OF RITZ SOLUTION
UNTAMPED SPHERE
UNTAMPED CUBE
UNTAMPED CYLINDER
UNTAMPED SLAB
HALF-TAMPED SLAB
1.4
1.2
1.0

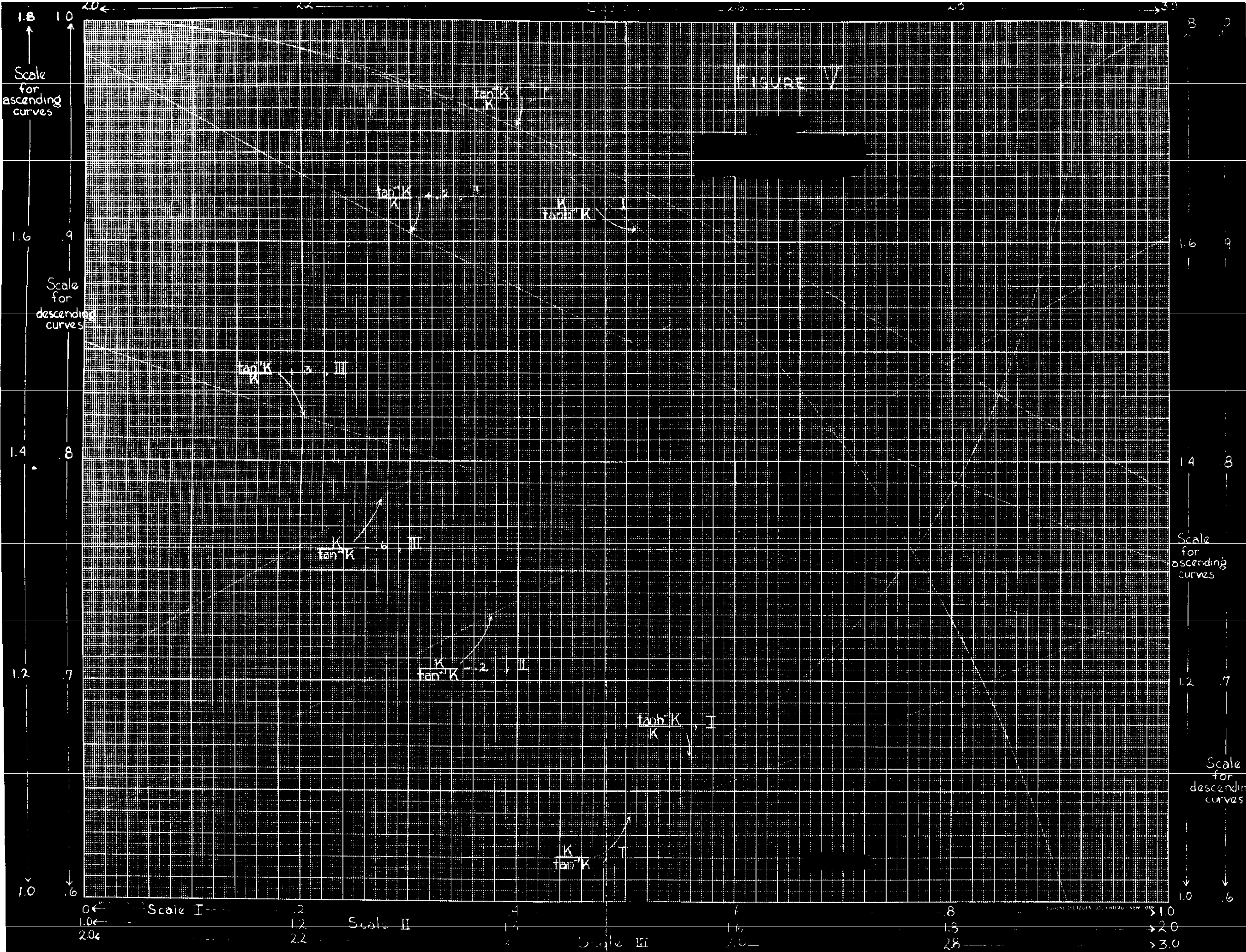
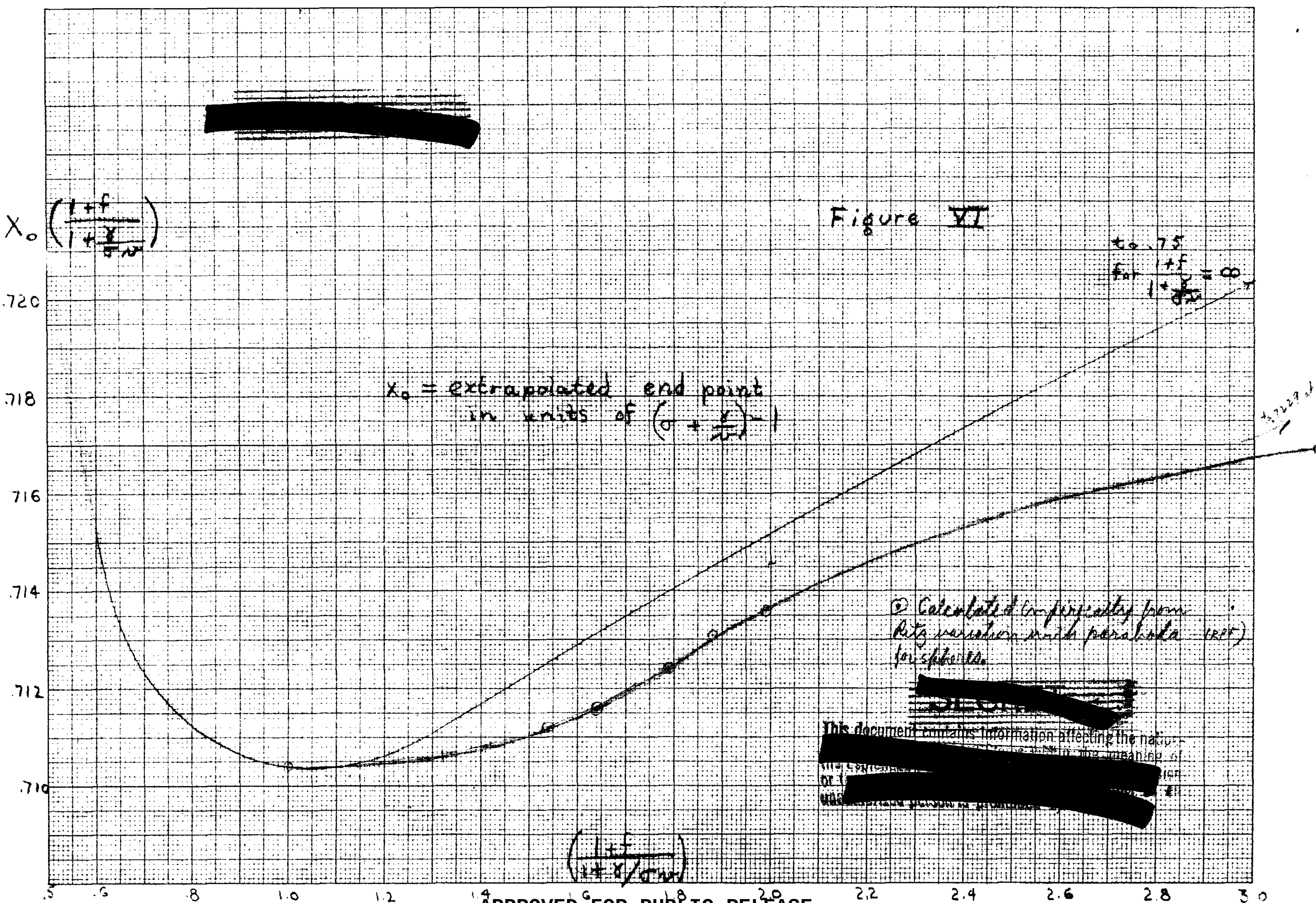


FIGURE V
[redacted]
Scale for ascending curves
Scale for descending curves
tan⁻¹K / K , I
tan⁻¹K / K , .2 , II
K / tan⁻¹K , I
tan⁻¹K / K , .3 , III
K / tan⁻¹K , .6 , III
K / tan⁻¹K , .2 , II
tan⁻¹K / K , I
K / tan⁻¹K , I
Scale I
Scale II
Scale III
[redacted]



[redacted]
Figure VI
$X_0\left(\frac{1+f}{1+\frac{\gamma}{\sigma\nu}}\right)$
X_0 = extrapolated end point in units of $\left(\sigma+\frac{\gamma}{\nu}\right)^{-1}$
.720
.718
.716
.714
.712
.710
.5
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
$\left(\frac{1+f}{1+\gamma/\sigma\nu}\right)$
⊙ Calculated empirically from Ritz variation with parabola for spheres.
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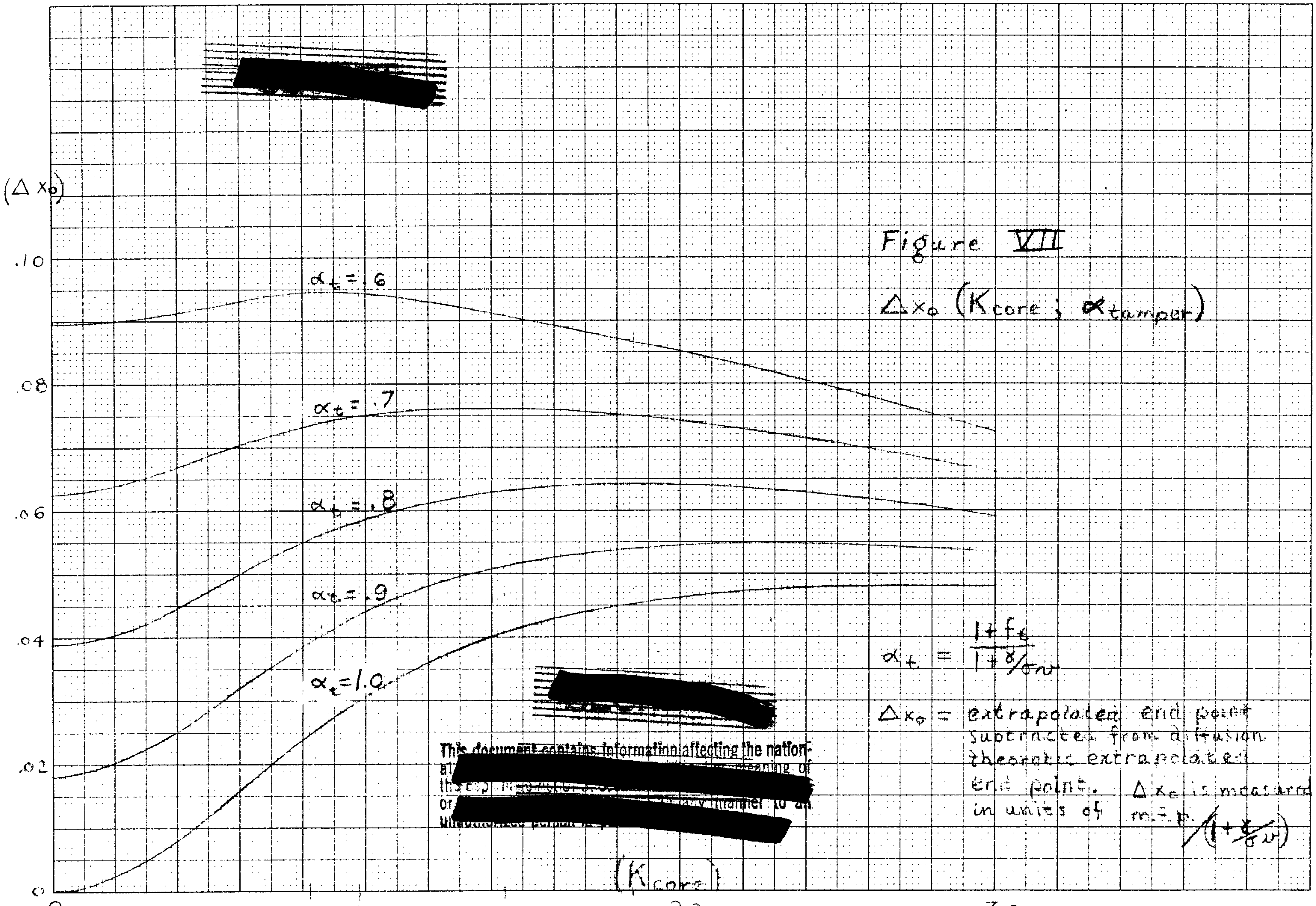
[redacted]
(Δx_o)
.10
.08
.06
.04
.02
0
$\alpha_t = .6$
$\alpha_t = .7$
$\alpha_t = .8$
$\alpha_t = .9$
$\alpha_t = 1.0$
Figure VII
$\Delta x_o \, (K_{core} \, ; \, \alpha_{tamper})$
$\alpha_t = \frac{1 + f_t}{1 + \gamma/\sigma\nu}$
Δx_o = extrapolated end point subtracted from diffusion theoretic extrapolated end point. Δx_o is measured in units of m.f.p./$(1 + \frac{\gamma}{\sigma\nu})$
[redacted]
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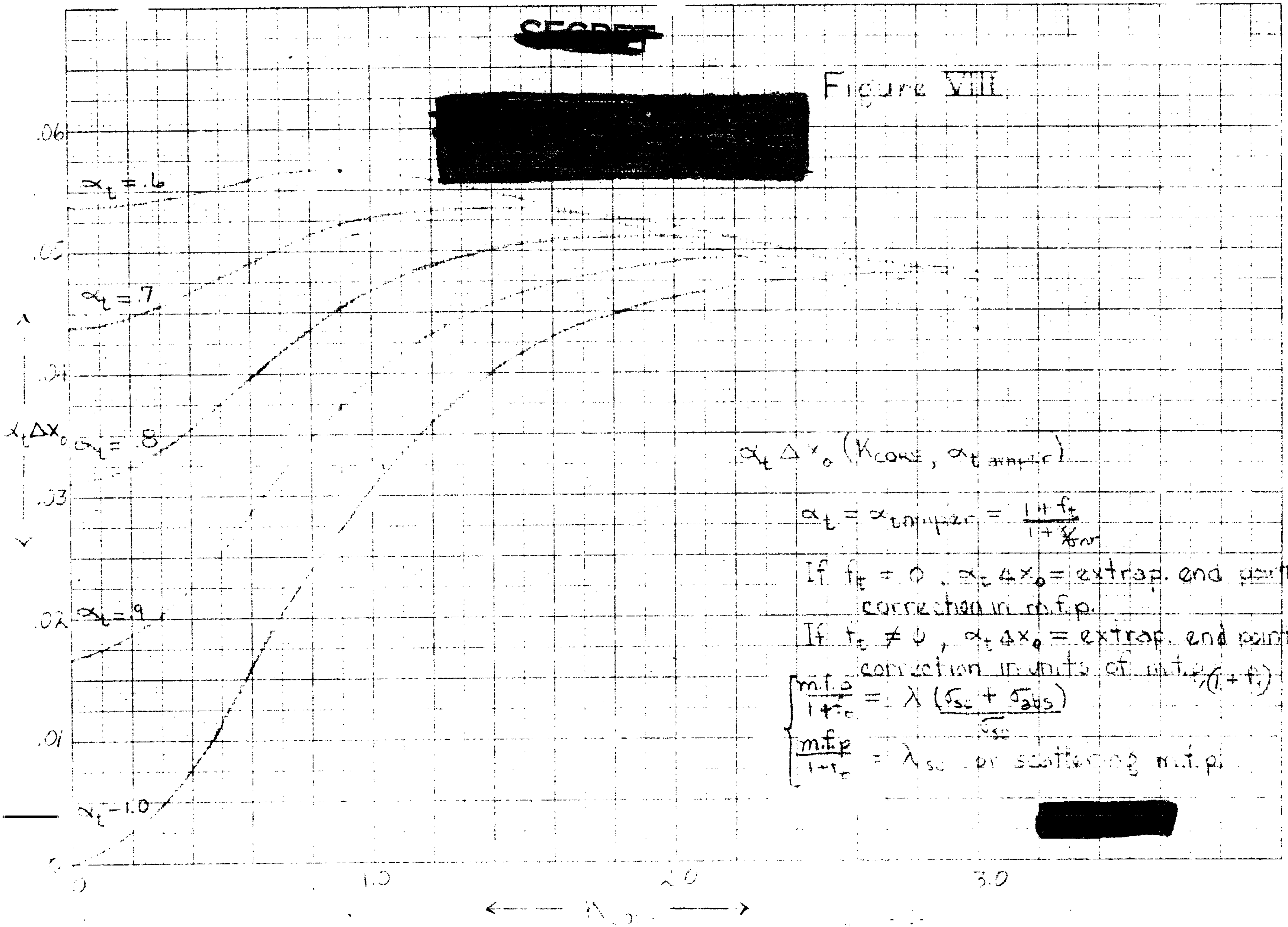
[redacted]
(K_{core})
0
1.0
2.0
3.0

[redacted]

Figure VIII

[redacted]



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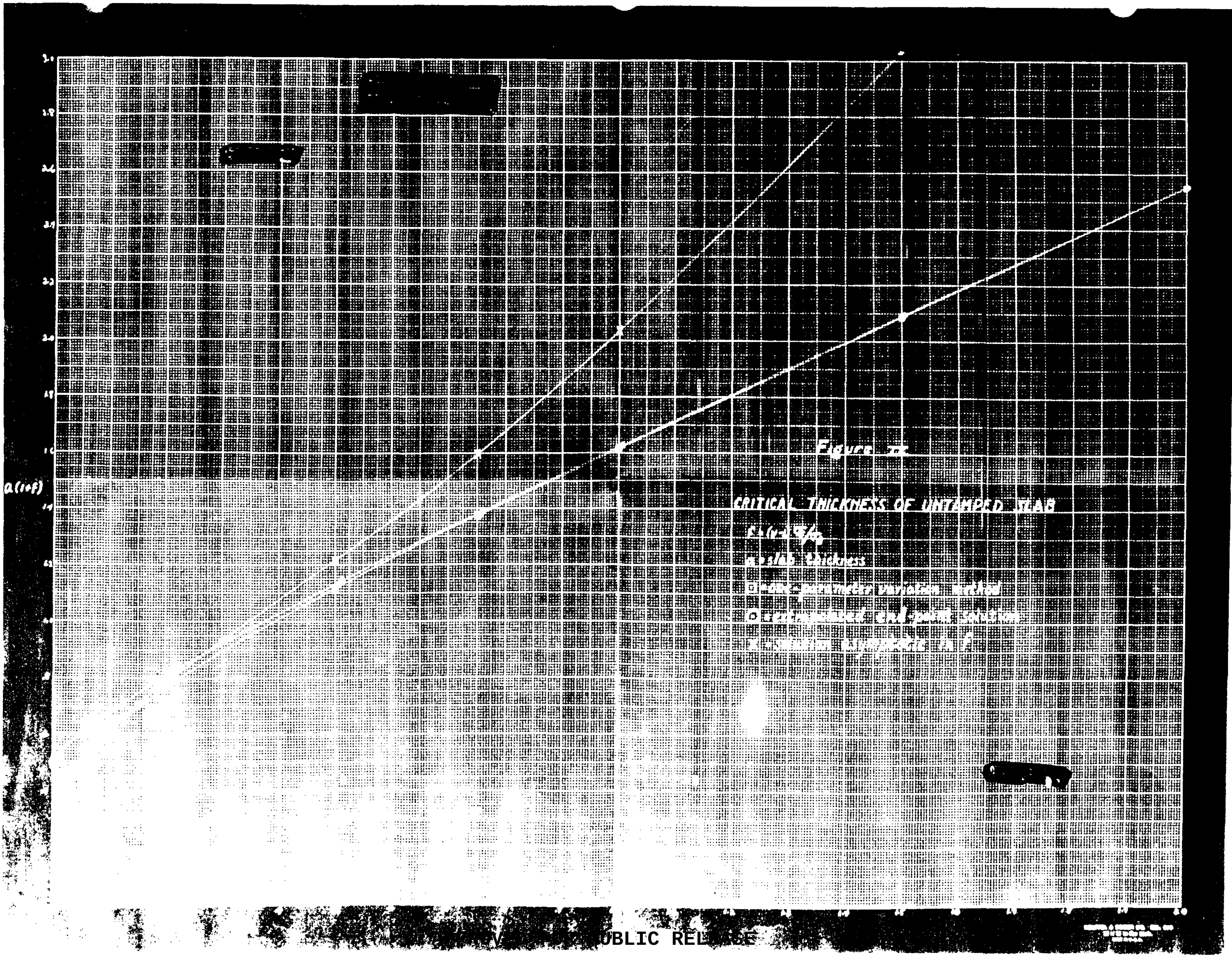

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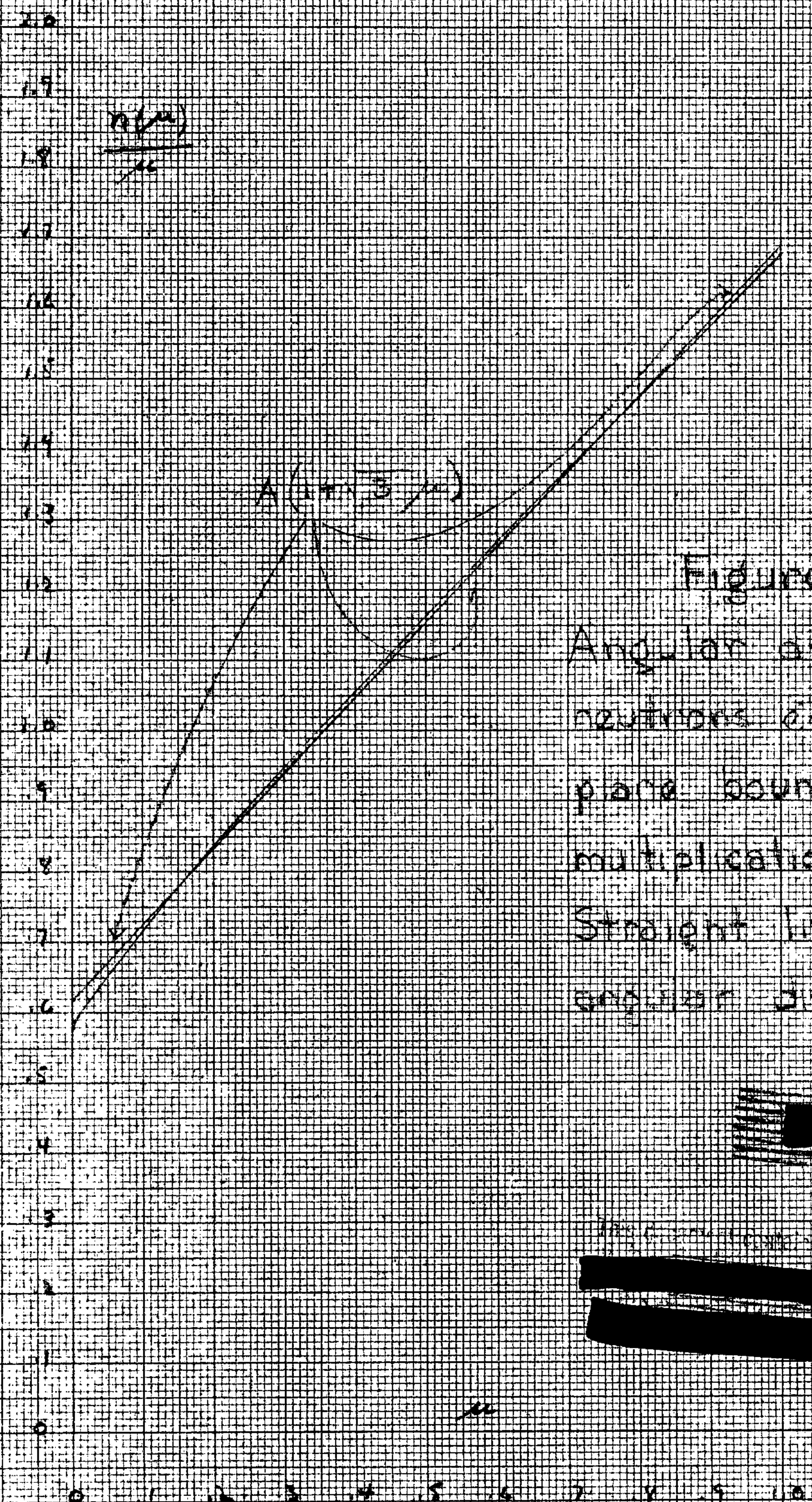

Figure X

Angular distribution of neutrons emerging at a plane boundary with no multiplication or absorption. Straight line is Fermi angular distribution.

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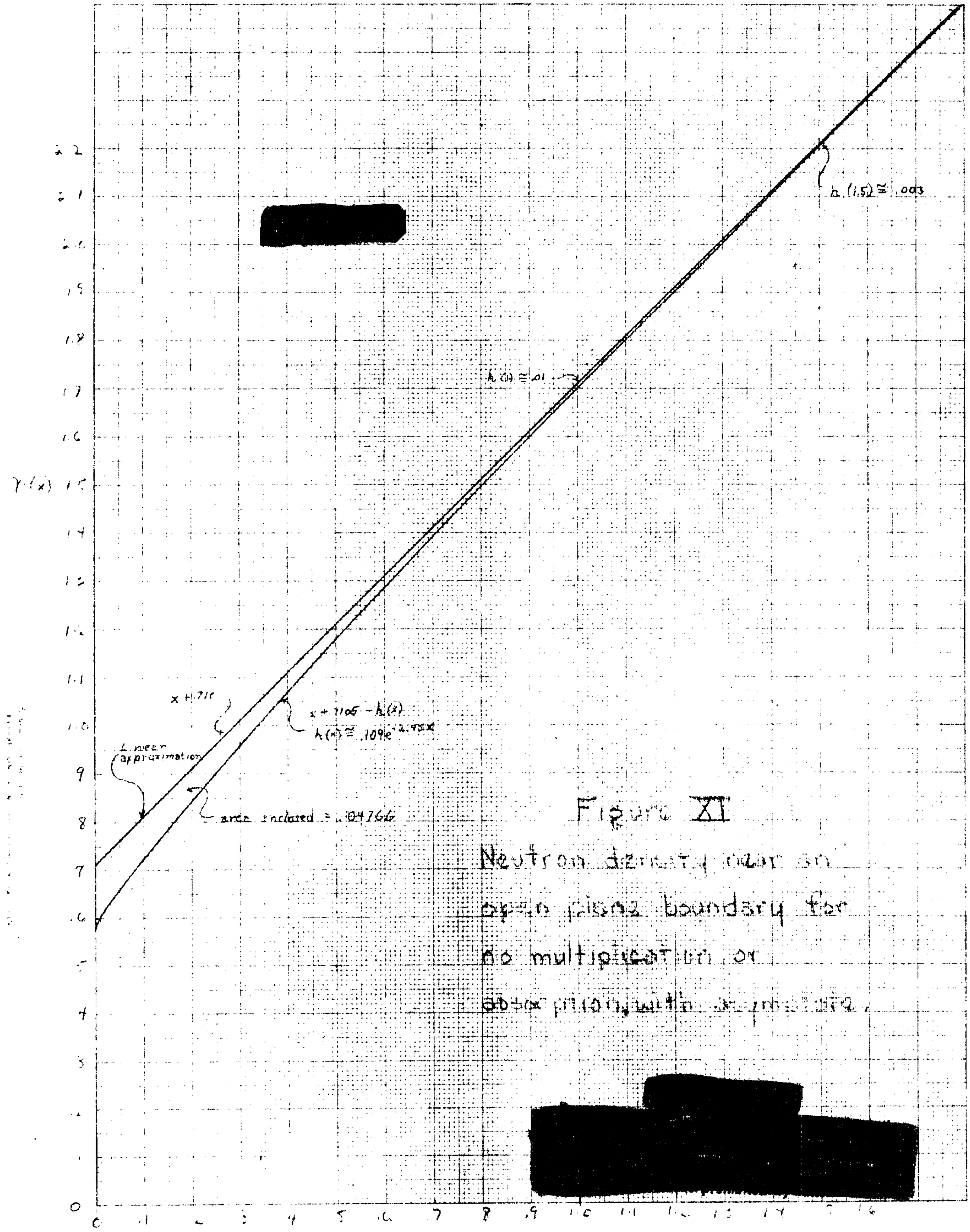
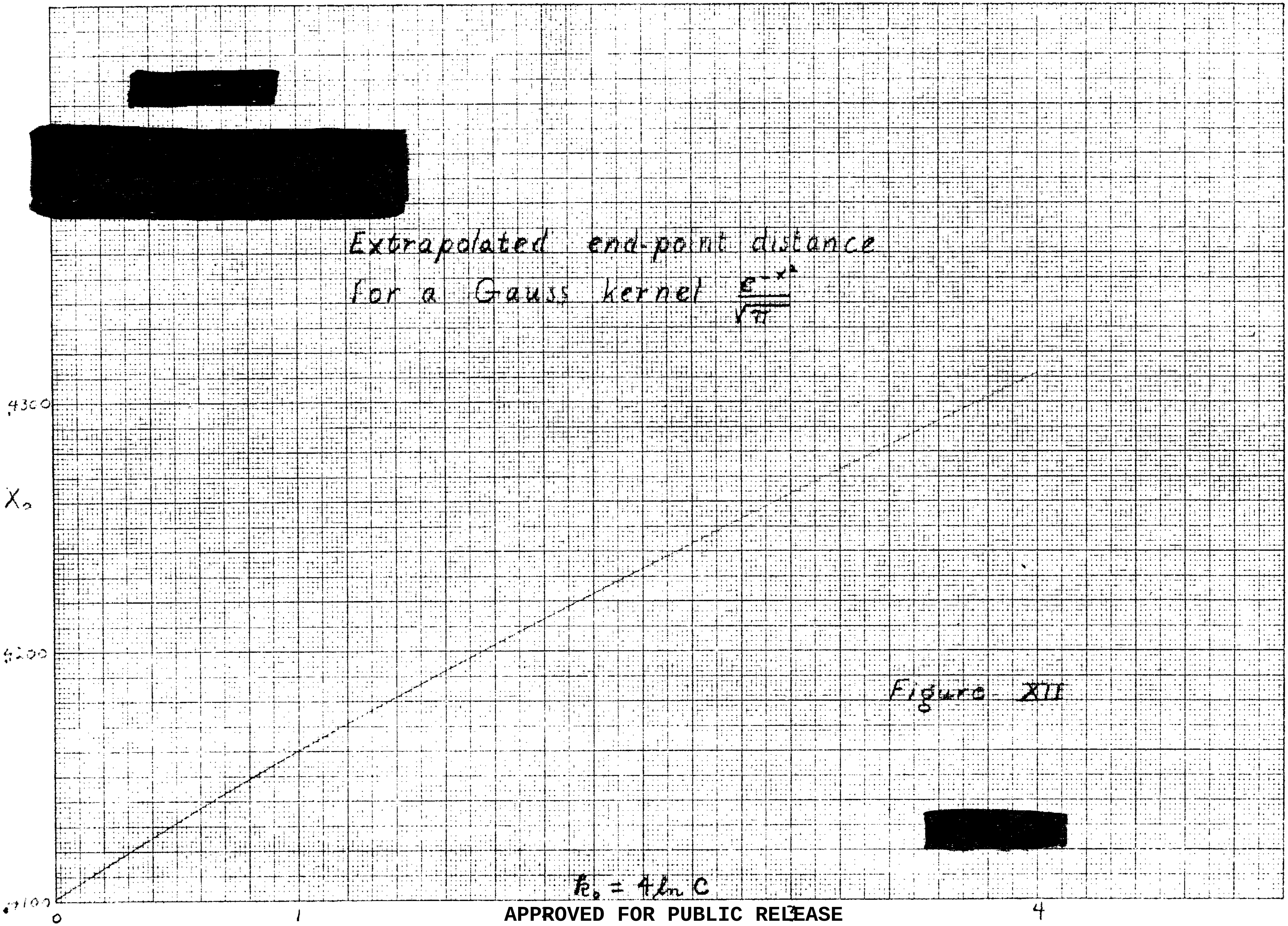


Figure XI

Neutron density near an open plane boundary for no multiplication or absorption, with [illegible].



[redacted]
[redacted]
Extrapolated end-point distance
for a Gauss kernel $\frac{e^{-x^2}}{\sqrt{\pi}}$
.4300
X_0
.4200
Figure XII
[redacted]
$k_0 = 4 \ln C$
.4100
0
1
4

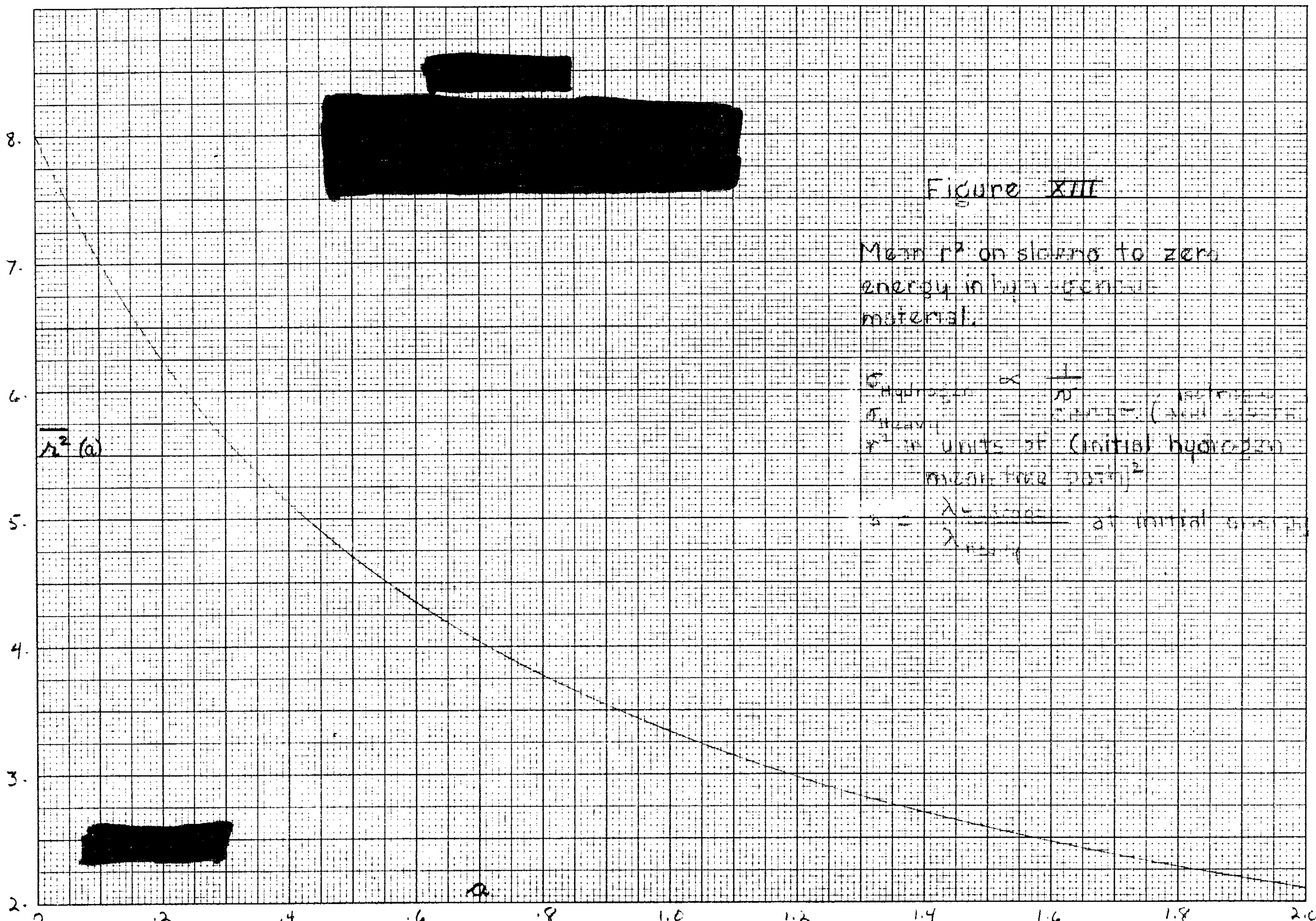

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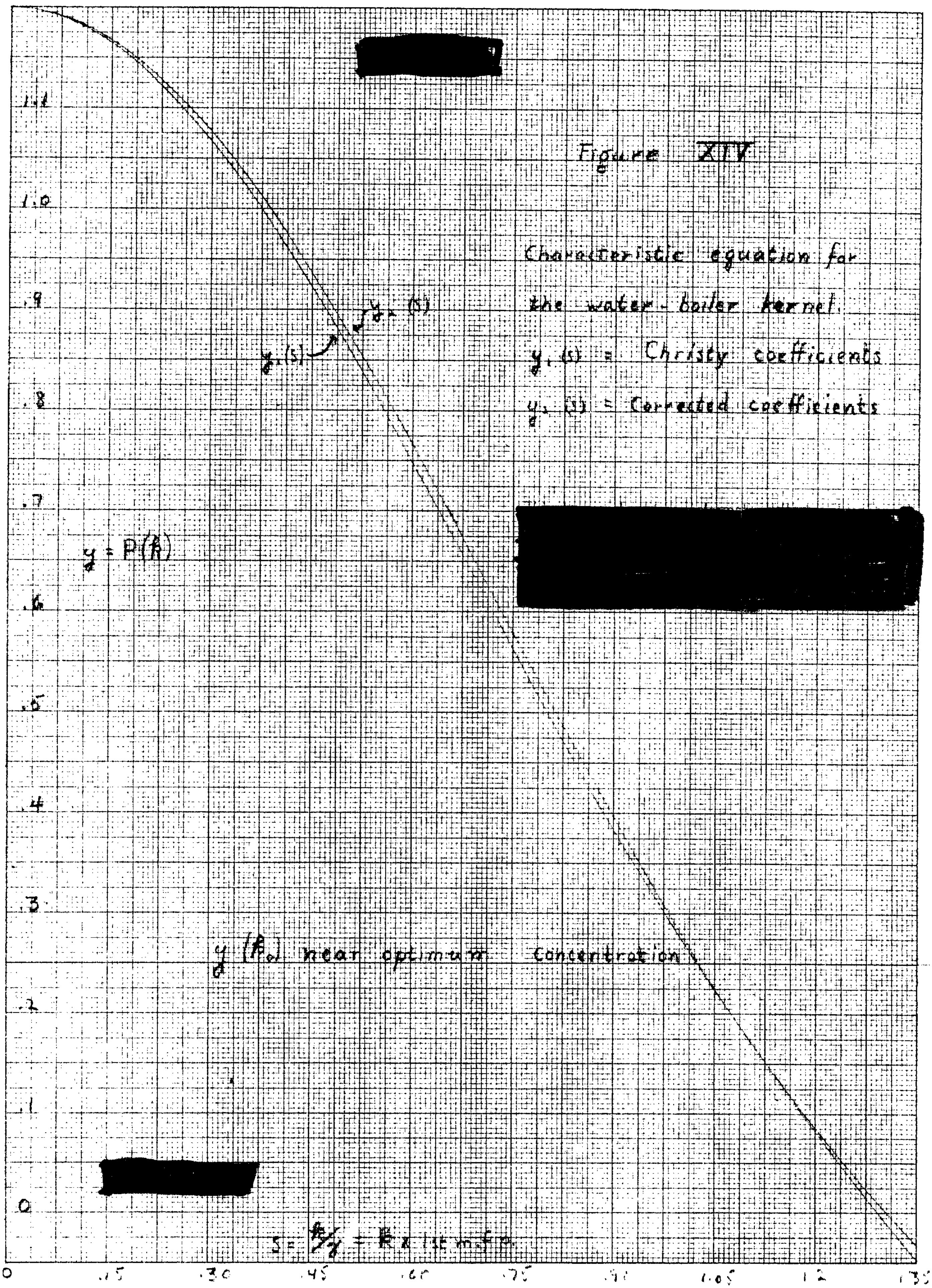
Figure XIII
Mean r² on slowing to zero
energy in hydrogenous
material.
∝
units of (initial hydrogen
mean free path)²
at initial energy
$\overline{r^2}$ (a)
8.
7.
6.
5.
4.
3.
2.
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
a
[redacted]



[redacted]
Figure XIV
Characteristic equation for the water-boiler kernel.
$y_1(s)$ = Christy coefficients
$y_2(s)$ = Corrected coefficients
[redacted]
$y_2(s)$
$y_1(s)$
$y = P(k)$
$y(k_0)$ near optimum concentration
[redacted]
$s = k/y = k \times 135$ m.f.p.
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
.15
.30
.45
.60
.75
.90
1.05
1.2
1.35

[redacted]

[redacted]

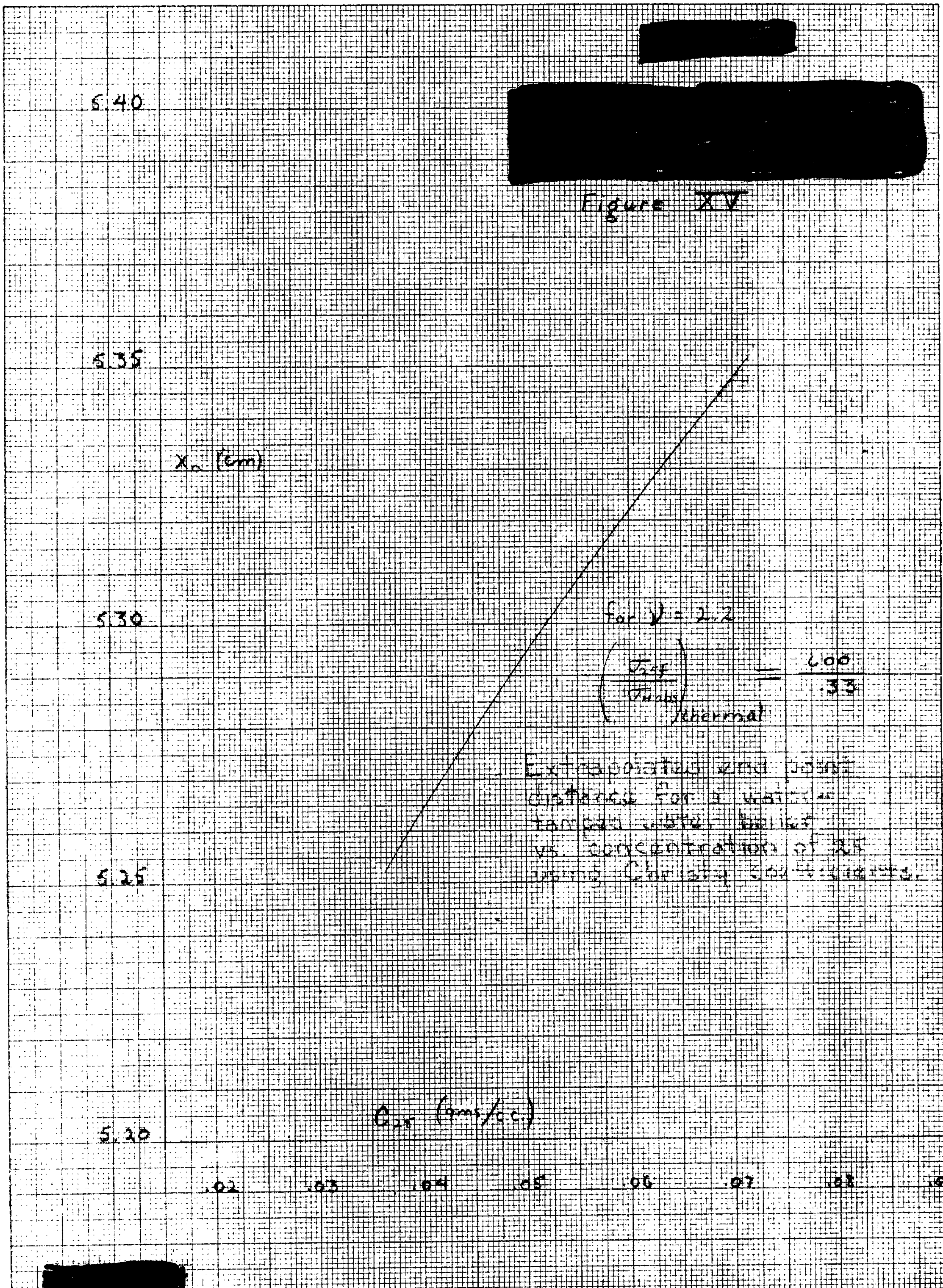


Figure XV

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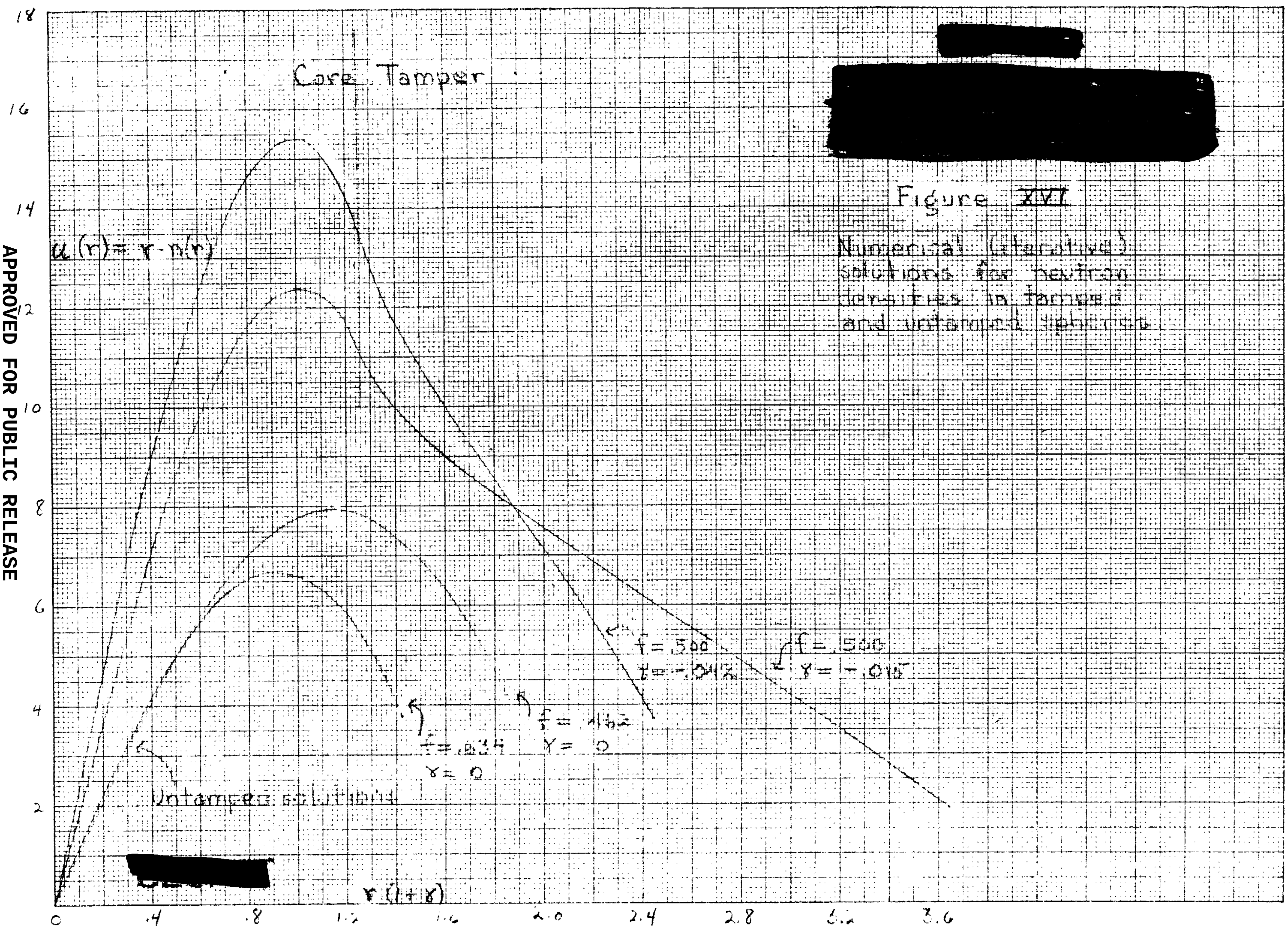

[redacted]

[redacted]

Figure XVI

Numerical (iterative) solutions for neutron densities in tamped and untamped spheres

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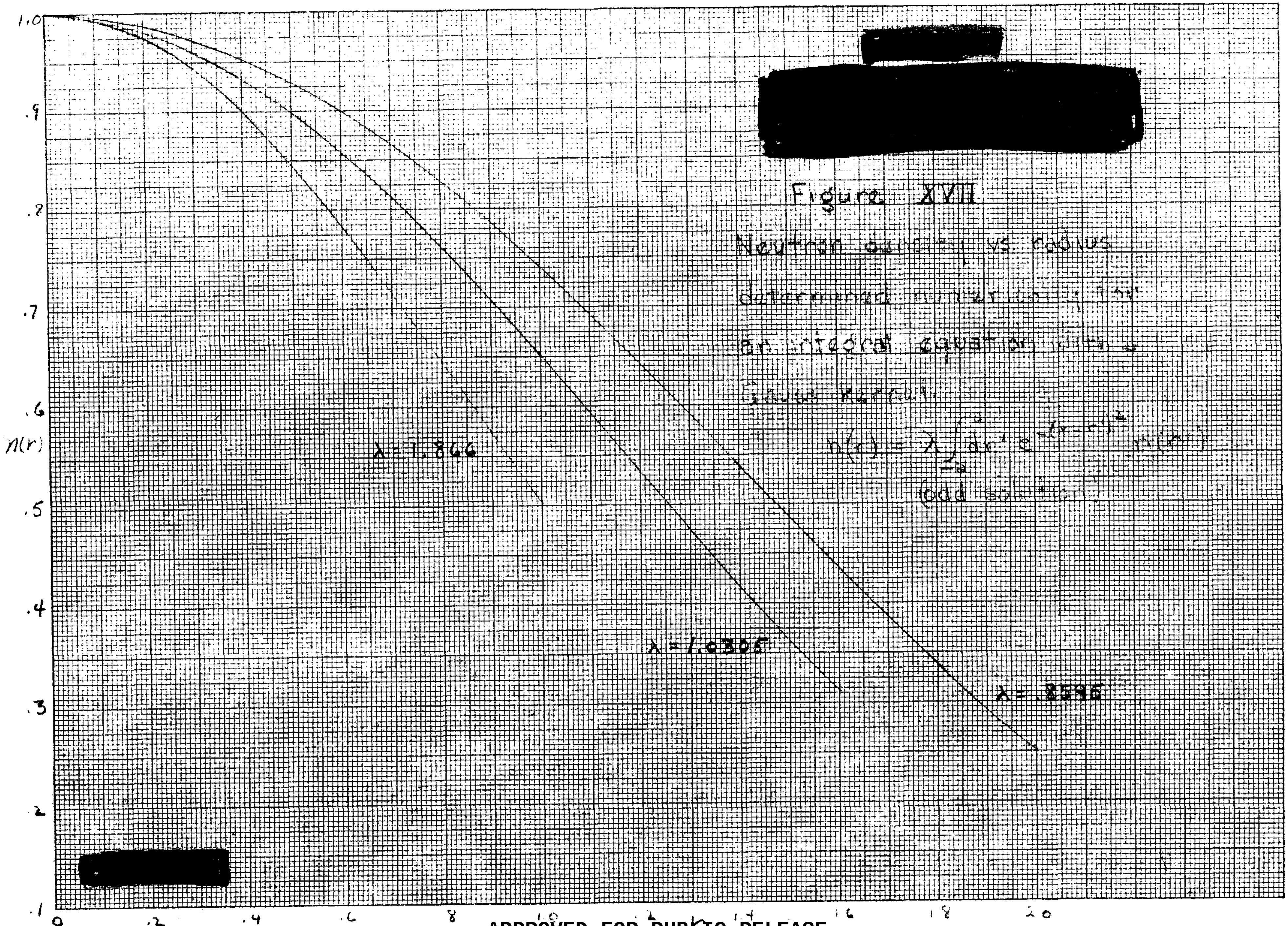

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Figure XVII

Neutron density vs radius determined numerically for an integral equation with a Gauss kernel.

$$n(r) = \lambda \int_{-a}^{a} dr' \, e^{-(r-r')^2} \, n(r')$$

(odd solution)

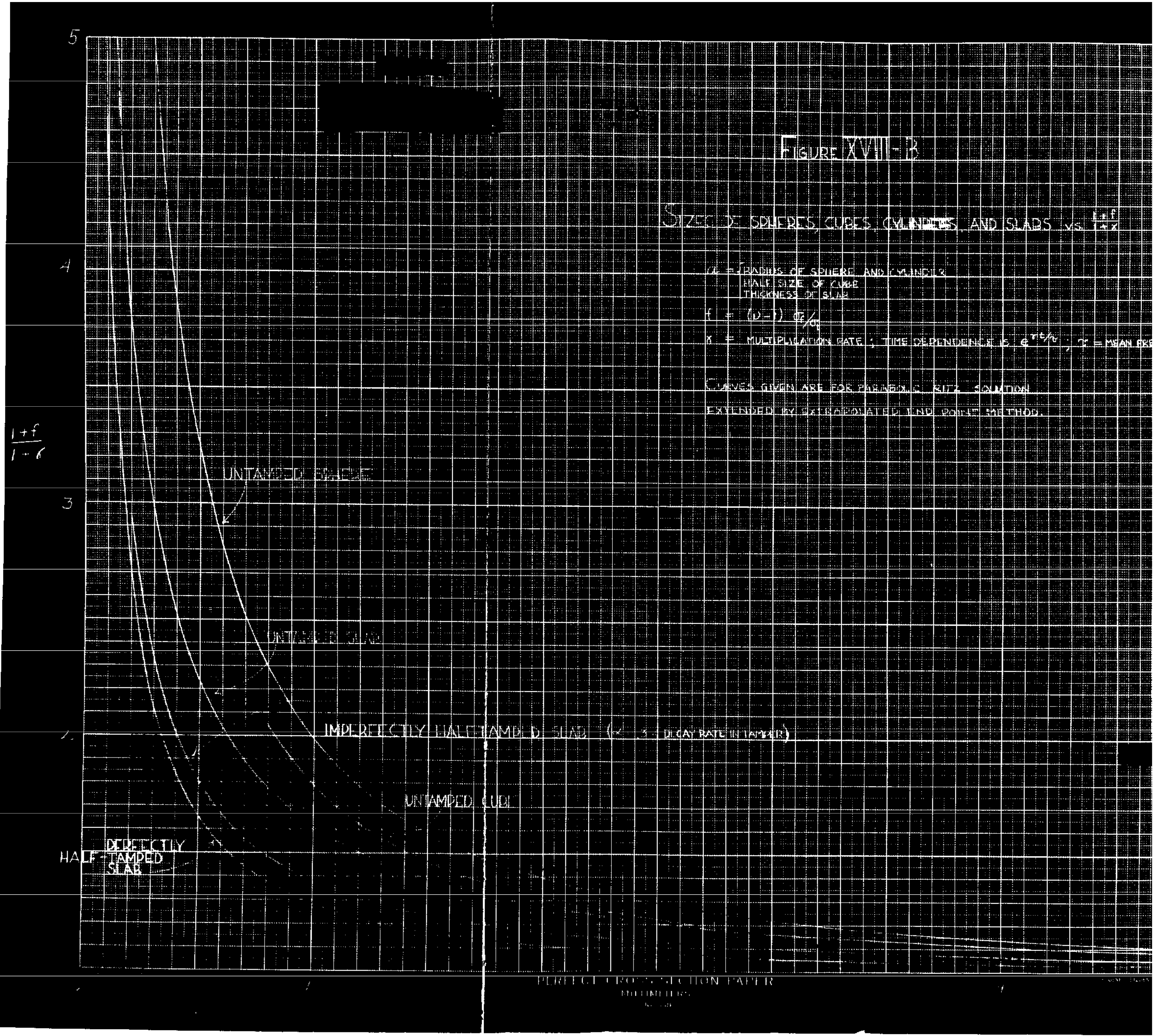
FIGURE XVIII-B
SIZES OF SPHERES, CUBES, CYLINDERS, AND SLABS vs. $\frac{1+f}{1+\gamma}$
a = RADIUS OF SPHERE AND CYLINDER
HALF SIZE OF CUBE
THICKNESS OF SLAB
f = (ν−1) σ$_f$/σ$_t$
γ = MULTIPLICATION RATE ; TIME DEPENDENCE IS $e^{\gamma t/\tau}$; τ = MEAN FRE
CURVES GIVEN ARE FOR PARABOLIC RITZ SOLUTION
EXTENDED BY EXTRAPOLATED END POINT METHOD.
UNTAMPED SPHERE
UNTAMPED SLAB
IMPERFECTLY HALF-TAMPED SLAB
DECAY RATE IN TAMPER
UNTAMPED CUBE
PERFECTLY
HALF-TAMPED
SLAB
5
4
3
$\frac{1+f}{1+\gamma}$
PERFECT CROSS SECTION PAPER
MILLIMETERS

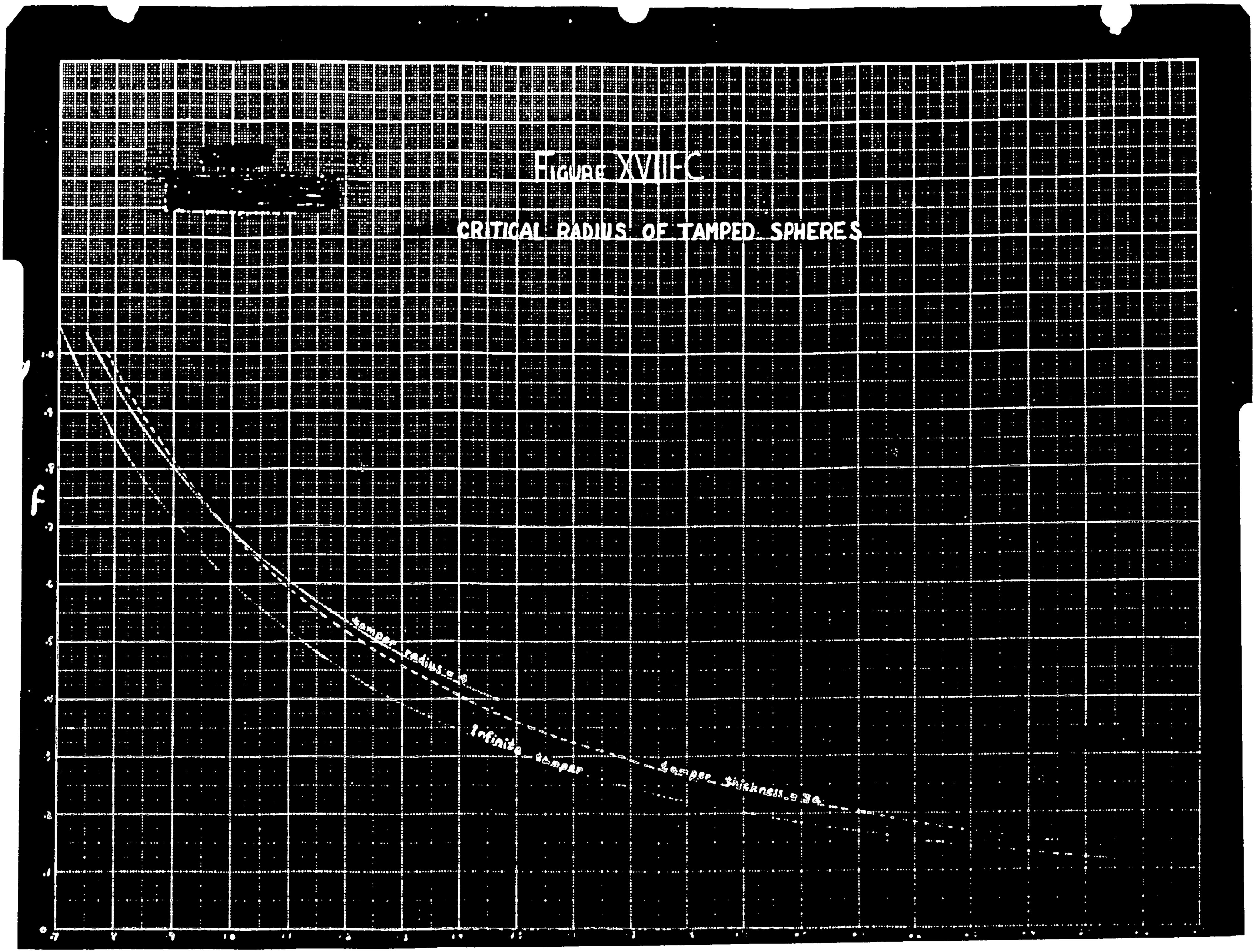
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Figure XVIII C
CRITICAL RADIUS OF TAMPED SPHERES
f
tamper radius
Infinite tamper
tamper thickness

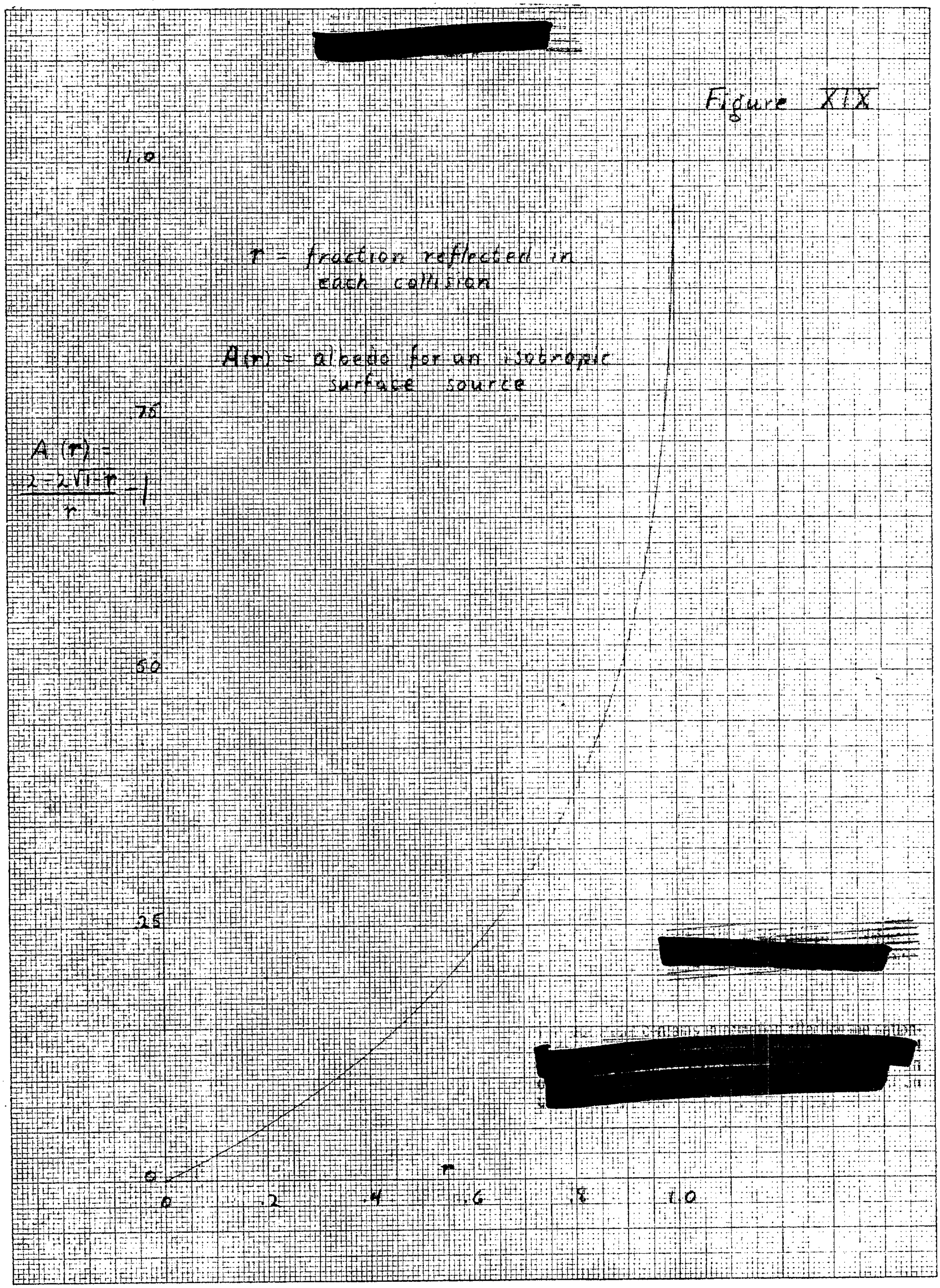

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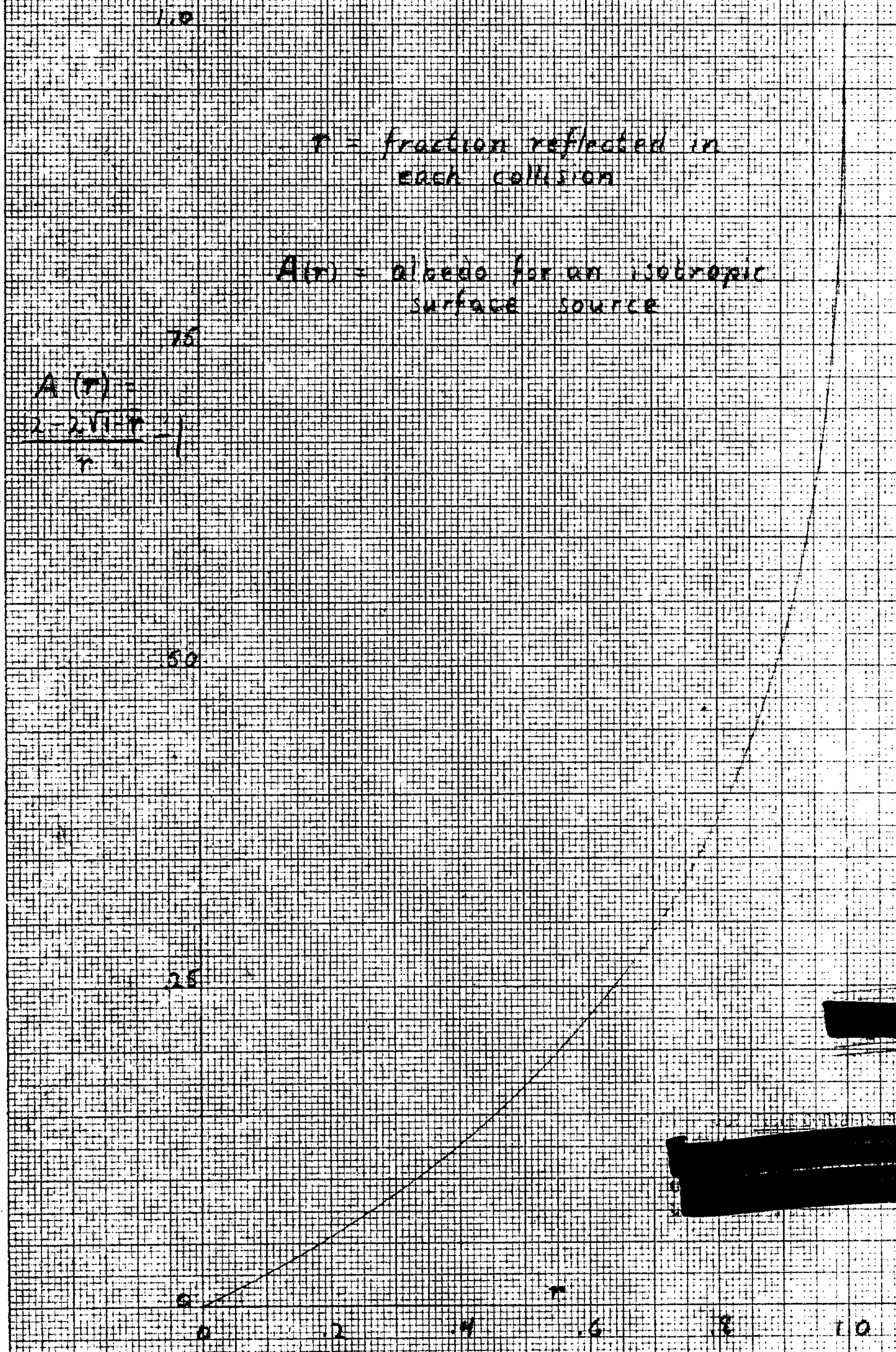
Figure XIX



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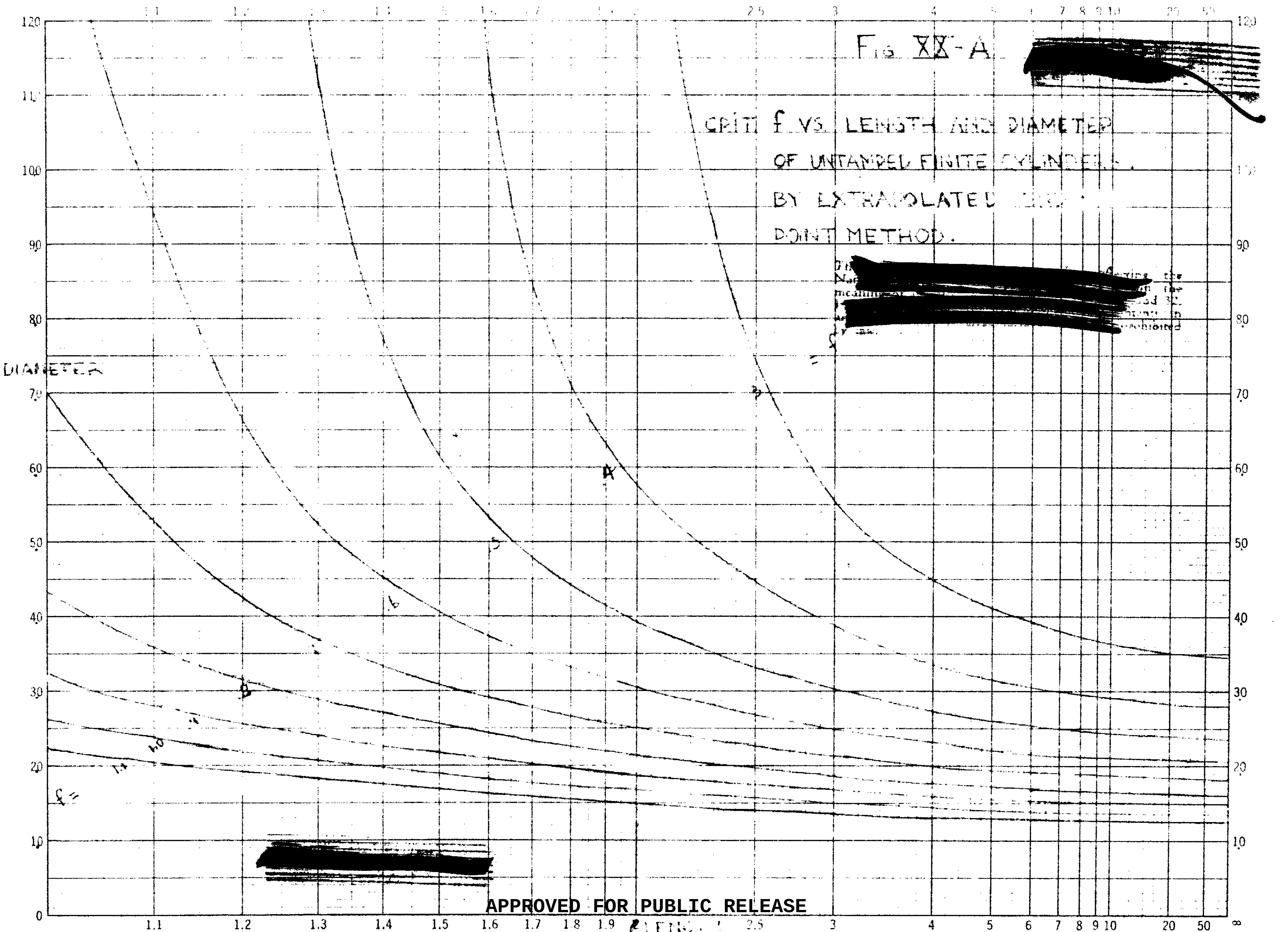
Fig XX-A
[redacted]
CRITI f VS. LENGTH AND DIAMETER
OF UNTAMPED FINITE CYLINDE
BY EXTRAPOLATED
POINT METHOD.
[redacted]
DIAMETER
f =
A
B
[redacted]
120
110
100
90
80
70
60
50
40
30
20
10
0
1.1
1.2
1.3
1.4
1.5
1.6
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1.8
1.9
2.5
3
4
5
6
7
8
9
10
20
50
∞

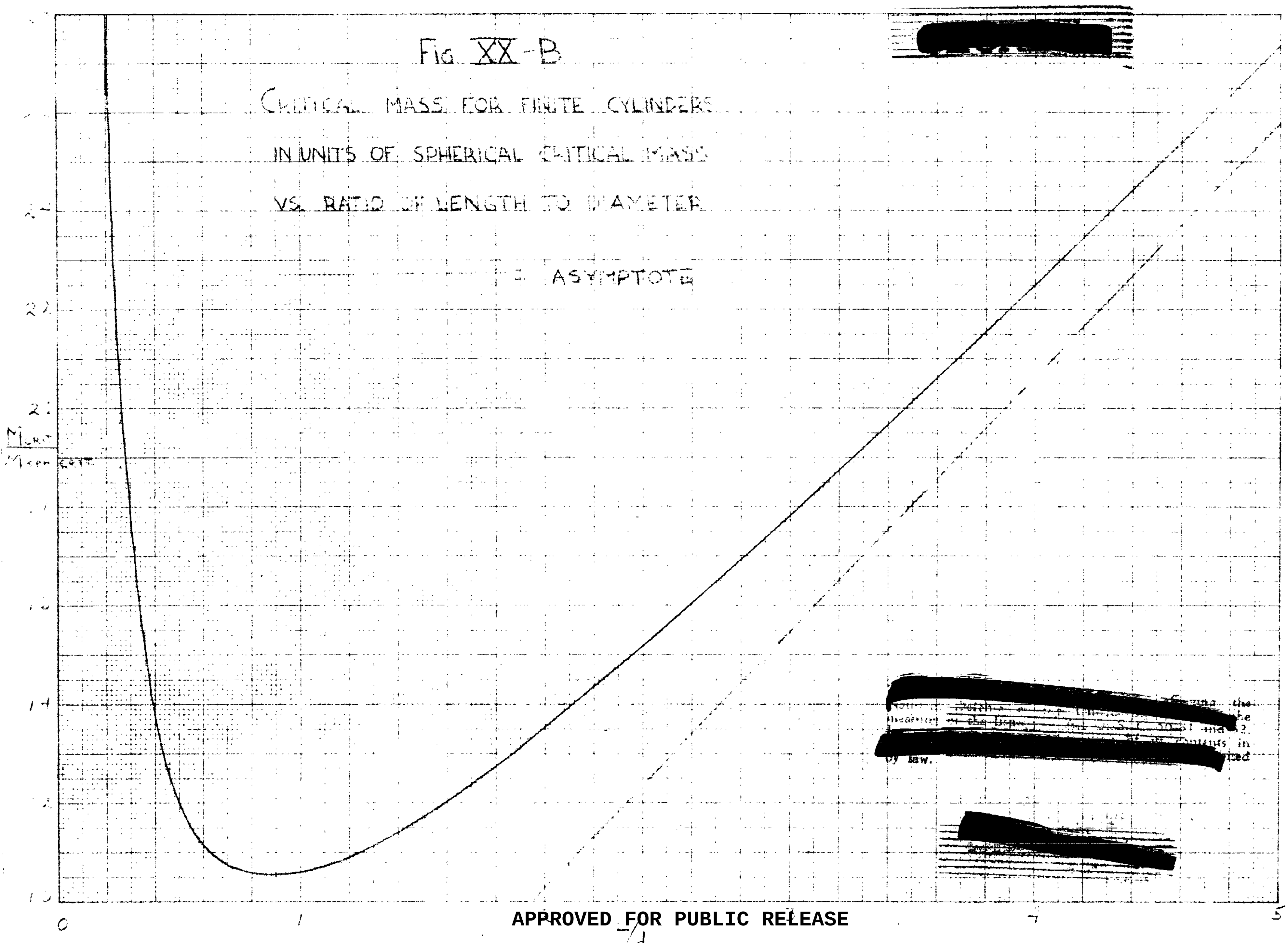
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Fig. XX-B
CRITICAL MASS FOR FINITE CYLINDERS
IN UNITS OF SPHERICAL CRITICAL MASS
VS. RATIO OF LENGTH TO DIAMETER
ASYMPTOTE
0
1
4
5
[redacted]
[redacted]

[redacted]

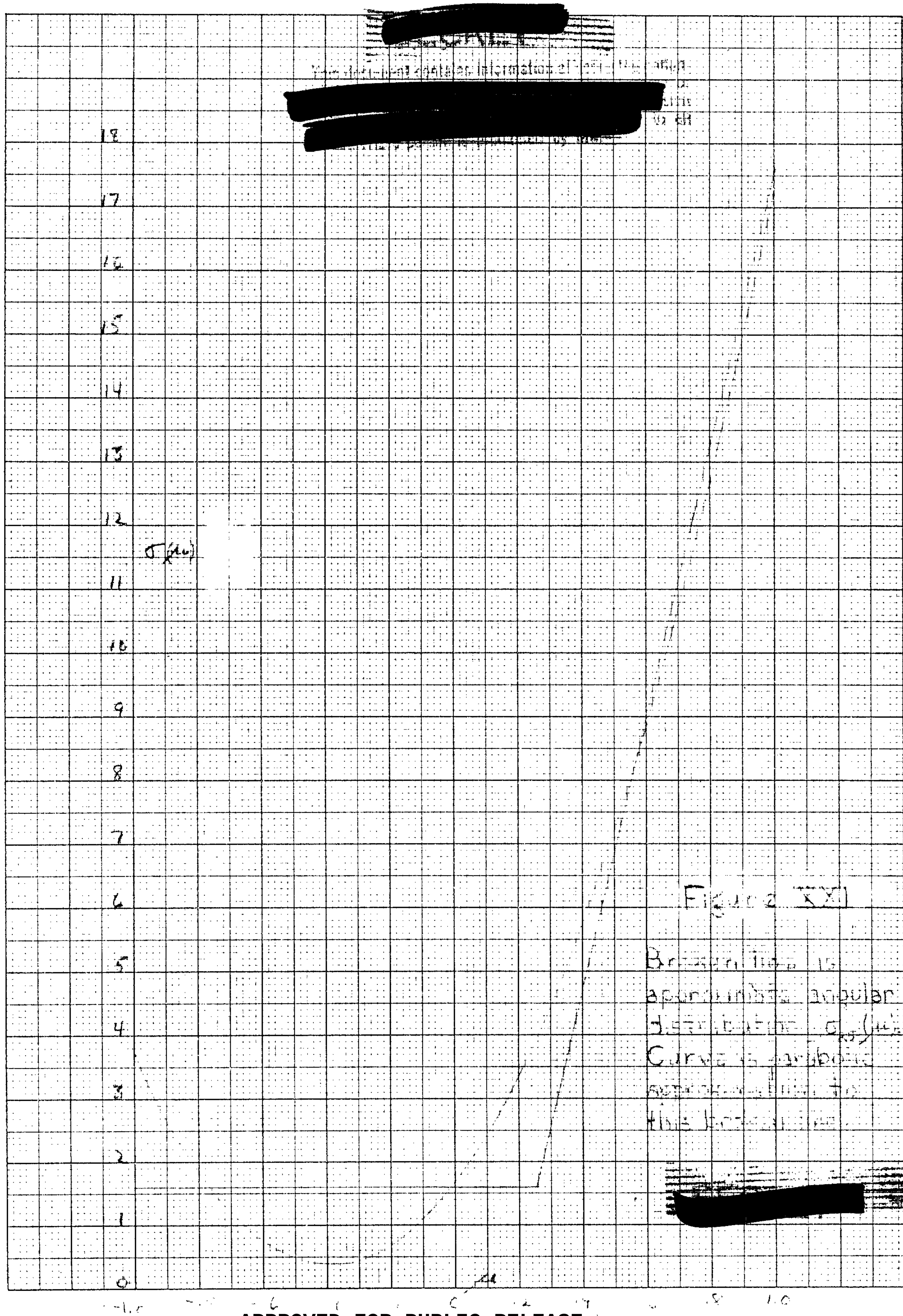

Figure XXI

Broken line is approximate angular distribution $\sigma_{el}(\mu)$. Curve is parabolic approximation to this distribution.

[redacted]

10 × 10 to the half inch, 5th lines accented.
Engraving, 7 × 10 in.
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